MATH 590: QUIZ 10 SOLUTIONS

Find the singular value decomposition of $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ by following the steps below. **Please indicate clearly each step in your solution**. Each step is worth 2 points.

1. Calculate A^tA and its characteristic polynomial $p_{A^tA}(x)$.

Solution. $p_{A^tA}(x) = \begin{vmatrix} x-1 & 1 & 0 \\ 1 & x-1 & 0 \\ 0 & 0 & x-4 \end{vmatrix} \{(x-1)^2 - 1\}(x-4) = x(x-4)(x-2).$

2. Find the non-zero eigenvalues of A^tA: $\lambda_1 > \lambda_2 > 0$.

Solution. $\lambda_1 = 4, \lambda_2 = 2$.

3. Find: A unit eigenvector u_1 of λ_1, a unit eigenvector u_2 for λ_2 and a unit vector u_3 such that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.

Solution For $E_4 =$ null space of $\begin{pmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{EROs}}{\to} \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so we take $u_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

For $E_2 =$ null space of $\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \overset{\text{EROs}}{\to} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so we take $u_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

Since A^tA is symmetric, with distinct eigenvalues, the unit eigenvectors for each eigenvalue will be an orthonormal basis for $\mathbb{R}^3$, so we for u_3, a unit vector generating $E_0 =$ null space of $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \overset{\text{EROs}}{\to}$ $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so $u_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Thus, we have an orthonormal basis for $\mathbb{R}^3$.

4. Set $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $v_1 = \frac{1}{\sigma_1} Au_1$, and $v_2 = \frac{1}{\sigma_2} Au_2$. Show that v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.

Solution. $\sigma_1 = \sqrt{4} = 2, \sigma_2 = \sqrt{2}, v_1 = \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Clearly v_1, v_2 is an orthonormal basis for $\mathbb{R}^2$.

5. Let P be the orthogonal matrix whose columns are u_1, u_2, u_3, Q the orthogonal matrix whose columns are v_1, v_2, and $\sum = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$. Verify that $A = Q \sum P^t$.

Solution. $AP = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 2 & 0 & 0 \end{pmatrix}$.

$Q \sum = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 2 & 0 & 0 \end{pmatrix}$. Thus, $AP = Q \sum$, so $A = Q \sum P^t$, since $P^{-1} = P^t$.